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UNIVERSAL EQUATION OF TRANSIENT PLANE JET IN CONCURRENT STREAM

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An equation is derived and subsequently integrated which is "universal" not only with respect to velocity of the concurrent stream and initial conditions of jet discharge but also with respect to choice of characteristic scale for the transverse coordinate.

A universal equation for a transient laminar jet and a transient gradiental concurrent stream of incompressible fluid has been derived in an earlier study [1] without the use of any integral relations, i.e., in purely differential form. We will now write this equation and the boundary conditions for the dimensionless flow function φ in the form

$$B^2\frac{\partial^3\varphi}{\partial\eta^3}-(r_{01}+f_{00}r_{10}+f_{10})\frac{\partial\varphi}{\partial\eta}+\left(\frac{g_{01}+f_{00}g_{10}}{2}+f_{10}\right)\eta\frac{\partial^2\varphi}{\partial\eta^2}-$$
$$-r_{10}\left(\frac{\partial\varphi}{\partial\eta}\right)^2+\left(\frac{g_{10}}{2}+r_{10}\right)\varphi\frac{\partial^2\varphi}{\partial\eta^2}=\sum_{k=n=i=j=l=m=0}^{\infty}\left[-\frac{\partial^2\varphi}{\partial f_{kn}\partial\eta}K+\right.$$
$$+\frac{\partial^2\varphi}{\partial r_{ij}\partial\eta}L+\frac{\partial^2\varphi}{\partial g_{lm}\partial\eta}M+\left(f_{00}+\frac{\partial\varphi}{\partial\eta}\right)\left(\frac{\partial^2\varphi}{\partial f_{kn}\partial\eta}N+\frac{\partial^2\varphi}{\partial r_{ij}\partial\eta}P+\right.$$
$$\left.\left.+\frac{\partial^2\varphi}{\partial g_{lm}\partial\eta}Q\right)-\frac{\partial^2\varphi}{\partial\eta^2}\left(\frac{\partial\varphi}{\partial f_{kn}}N+\frac{\partial\varphi}{\partial r_{ij}}P+\frac{\partial\varphi}{\partial g_{lm}}Q\right)\right],\qquad(1)$$
$$\varphi=\frac{\partial^2\varphi}{\partial\eta^2}=0\ \text{for}\quad\eta=0;\quad\frac{\partial\varphi}{\partial\eta}=0\ \text{for}\quad\eta\to\infty;$$

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$$\varphi = \varphi^0(\eta) \text{ for } f_{kn} = r_{ij} = g_{lm} = 0 \ (j \neq 0;\ m \neq 0);$$
$$r_{i0} = r_{i0}^0 = \text{const}, \ g_{l0} = g_{l0}^0 = \text{const}, \tag{2}$$

where $\varphi^0(\eta)$ is the self-adjoint Schlichting solution for a plane inundated jet.

In Eq. (1) has been used the following notation:

$$\varphi(\eta, f_{kn}, r_{ij}, g_{lm}) = \frac{B\psi}{u_{1m}h}; \ \eta = \frac{By}{h}, \tag{3}$$

$$\begin{aligned}
K &= [(k-1)r_{01} + (k+n)g_{01}]f_{kn} + f_{k,n+1},\\
L &= [(i-1)r_{01} + (i+j)g_{01}]r_{ij} + r_{i,j+1},\\
M &= [lr_{01} + (l+m-1)g_{01}]g_{lm} + g_{l,m+1},\\
N &= [(k-1)r_{10} + (k+n)g_{10}]f_{kn} + f_{k+1,n},\\
P &= [(i-1)r_{10} + (i+j)g_{10}]k_{ij} + r_{i+1,j},\\
Q &= [lr_{10} + (l+m-1)g_{10}]g_{lm} + g_{l+1,m}.
\end{aligned} \tag{4}$$

The series of parameters introduced to replace the longitudinal coordinate x and time t are

$$\left.\begin{aligned}
f_{kn} &= u_{1m}^{k-1}\frac{\partial^{k+n}U}{\partial x^k \partial t^n} z^{k+n},\\
r_{ij} &= u_{1m}^{i-1}\frac{\partial^{i+j}u_{1m}}{\partial x^i \partial t^j} z^{i+j},\\
g_{lm} &= u_{1m}^{l}\frac{\partial^{l+m}z}{\partial x^l \partial t^m} z^{l+m-1},
\end{aligned}\right\} (k,\ n,\ i,\ j,\ l,\ m = 0,\ 1,\ 2,\ \ldots) \tag{5}$$

where $z = h^2/\nu$.

These parameters, containing not only the velocity U of the concurrent stream and the excess velocity u_{1m} at the jet axis as well as their derivatives but also the quantity z related to some characteristic jet thickness and the derivatives of z, reflect the dependence of the jet development on the characteristics of the concurrent stream and on the initial conditions of jet discharge defining the individuality of a jet. Treatment of these parameters as variables has eliminated, in explicit form, the velocity of the concurrent stream as well as the quantities defining the individuality of a jet from Eq. (1) and the boundary conditions (2). This is what makes the equation universal. A numerical solution for a "segment" of this equation, obtained once and for all with only a limited number of parameters retained, will yield a set of velocity profiles for various combinations of their values. An analysis of the velocity profiles makes it possible to determine how their form depends on these parameters, which is of considerable interest. The velocity profiles can also be used for approximate solution of particular problems with the aid of integral equations, for determining the characteristic jet width and the axial velocity as functions of time and the longitudinal coordinate when a specific velocity distribution in the concurrent stream is given, also for determining the initial conditions.

For integrating Eq. (1), one must first determine the numerical value of the normalizing constant B. Its value depends on the choice of scale h(x, t) for the transverse coordinate [1]. The characteristic values of this scale, which also simplify subsequent calculations, can be found from the integral equations such as those of momentum and energy of a jet in a concurrent stream. Scales of this kind are [1]

$$\delta_1 = \frac{h}{B}\int_0^\infty \left(f_{00} + \frac{\partial\varphi}{\partial\eta}\right)\frac{\partial\varphi}{\partial\eta}\,d\eta; \quad \delta_2 = \frac{h}{B}\int_0^\infty \frac{\partial\varphi}{\partial\eta}\,d\eta;$$

$$\delta_3 = \frac{h}{B}\int_0^\infty \left(2f_{00} + \frac{\partial\varphi}{\partial\eta}\right)\frac{\partial\varphi}{\partial\eta}\,d\eta;$$

$$\delta_4 = \frac{h}{B}\int_0^\infty \left(f_{00} + \frac{\partial\varphi}{\partial\eta}\right)\left(2f_{00} + \frac{\partial\varphi}{\partial\eta}\right)\frac{\partial\varphi}{\partial\eta}\,d\eta. \tag{6}$$

It is well known that a favorable choice of one of these scales can play an important role in simplifying the solution of particular problems and in making their solutions more accurate, especially in the case of transient boundary layers of jets. The optimum scale can only be found, however, after the universal equation has been solved with various scales and the results compared. For this reason, integrating Eq. (1) with all scale (6) is very important.

The value of constant B for each scale can be found [1] from the self-adjoint Schlichting solution for an inundated jet [2]. We obtain B = 0.943 for $h = \delta_1 = \delta_3$, B = 1.414 for $h = \delta_2$, and B = 0.754 for $h = \delta_4$.

Accordingly, for each scale the universal equation (1) must be integrated with the corresponding value of constant B. It is possible to avoid this complication, however, and an equation can be obtained which is also universal with respect to choice of scale h.

Let B = 1. We will demonstrate that such a value of this constant corresponds to a scale equal to any of the characteristic thicknesses δ_i multiplied by some numerical factor. Indeed, in the case of a steady inundated jet [2] the characteristic thicknesses δ_i^0 can be expressed in the general form

$$\delta_i^0 = \beta_i \frac{\nu^{1/2}}{\alpha} x^{2/3} \,(\beta_1 = \beta_3 = 2;\ \beta_2 = 3;\ \beta_4 = 8/5). \tag{7}$$

We note that when $\beta_i = \beta_* = 3/\sqrt{2}$, the quantity $\delta_i^0 = \delta_*^0 = h_*$ selected as scale for the transverse coordinate will make B = 1 in accordance with the Schlichting solution. In order to establish the relation between δ_*^0 and the characteristic thickness δ_i^0, we let $\delta_*^0 = \varkappa_i \delta_i^0$ with $\varkappa_i$ standing for some numerical factor. Inserting this relation into experssion (7) with $\beta_i = \beta_* = 3/\sqrt{2}$ yields $\varkappa_i \delta_i^0 = \frac{3}{\sqrt{2}} \frac{\nu^{1/2}}{\alpha} x^{2/3}$, and thus $\delta_i^0 = \frac{3}{\varkappa_i \sqrt{2}} \frac{\nu^{1/2}}{\alpha} x^{2/3}$. Using the value of factor β_i for each thickness δ_i^0 according to relation (7) yields $\varkappa_1 = \varkappa_3 = 1.061$, $\varkappa_2 = 0.7071$, and $\varkappa_4 = 1.326$ respectively. It follows that one and the same value of the normalizing constant (B = 1) corresponds to any of the scale values

$$h_1 = h_3 = 1.061\delta_1 = 1.061\delta_3;\ h_2 = 0.7071\delta_2;\ h_4 = 1.326\delta_4. \tag{8}$$

In this way Eq. (1) appears to be universal not only with respect to velocity of the concurrent stream and the initial conditions of jet discharge but also with respect to scale h. It therefore can be integrated once and for all (in a given approximation) with a definite value of the normalizing constant (e.g., B = 1) common to various scales. It must be emphasized, however, that the scales $h = \delta_i$ under consideration here have been obtained with the aid of the integral equations for a jet in a concurrent stream and that, therefore, the arbitrariness of the choice of scale is, just as in the case of a boundary layer generally, limited by definite requirements.

In order to solve particular problems, it is necessary to know the values of the characteristic jet functions H_i and to successively stipulate $h_i = h_* = \varkappa_i \delta_i$ ($i = 1, 2, 3, 4$) as the scale. Calculations are then made according to the relation $H_i = \delta_i / h_* = \delta_i / \varkappa_i \delta_i = 1/\varkappa_i$. We obtain

$$H_1 = H_3 = 0.943;\ H_2 = 1.414;\ H_4 = 0.754. \tag{9}$$

In this way to each scale in series (8) there corresponds a numerical value of a characteristic function in series (9). The characteristic functions H_i not corresponding to the selected scale are variables depending on the parameters. All characteristic functions can be found after the universal equation has been solved, the solution being obtained, as has been noted earlier, independently of the choice of scale. This permits us to represent all H_i's as functions of the parameters. Only in the second stage of the process, solution of a particular problem, must one select a specific scale $h_i = \varkappa_i \delta_i$ from series (8) and equate the expression for H_i matching the selected scale to the corresponding constant in series (9). This equality is an equation and can be used, together with the integral equations of momentum and energy, for determining $u_{1m}(x, t)$ and $z(x, t)$ with $U(x, t)$ known a priori and with specific initial conditions stipulated. In these integral equations as well as in the expression for function H_i it is then necessary to explicate the parameters in accordance with

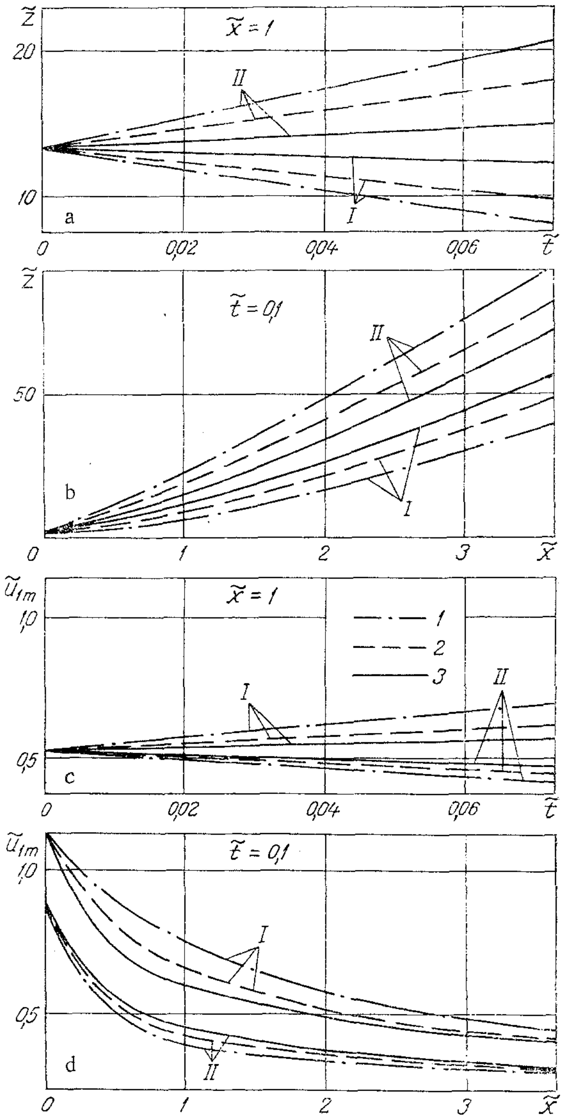

Fig. 1. Calculated characteristic jet thickness (a, b) and velocity at jet axis (c, d) as functions of time in fixed section (a, c) and as function of longitudinal coordinate at fixed instant (b, d) (all quantities dimensionless): 1) according to equations of momentum and energy, 2) according to equation of momentum and expansion of H_1 = const, 3) according to method [3]; I) buildup, II) decay.

the velocity of the concurrent stream U(x, t) known, one evaluates the parameters as functions of x and t, and thus also the velocity profiles in the jet as functions of these coordinates. The solution of the problem is then completed.

As the universal Eq. (1) was integrated, the flow function φ was sought in the form of a "segment" of a power series in the parameters, with first and second powers as well as derivatives of all initial parameters in series (5) retained. Upon insertion of the resulting polynomial into Eq. (1), in which the same parameters and their derivatives have been retained, and equating the coefficients of like monomial combinations of parameters, one obtains a system of ordinary differential equations in the coefficients of this polynomial. A numerical solution of these equations on a computer by the method of elimination yielded actual values of these coefficients (not shown here), whereupon the form of the characteristic functions H_i could be determined. For illustration, here are the expressions for the first two functions in the linear approximation:

$$H_1 = 0.943 + 4.827 f_{00} - 0.939 f_{10} - 5.70 (r_{10} - r_{10}^0) + 5.70 (g_{10} - g_{10}^0) + 0.573 f_{01} + 3.327 r_{01} + 4.938 g_{01},$$

$$H_2 = 1.414 - 1.276 f_{00} - 1.324 f_{10} - 5.074 (r_{10} - r_{10}^0) + 5.078 (g_{10} - g_{10}^0) + 0.818 f_{01} + 5.184 r_{01} + 5.329 g_{01}, \quad (10)$$

where parameter r_{10}^0 and g_{10}^0 correspond to the Schlichting solution for an inundated jet.

We note that, after integration of the universal equation, an analysis of the velocity profiles in a jet has revealed the following trends: a jet narrows during buildup and widens during decay; a concurrent stream narrows a jet, an accelerated stream more and a decelerated stream less.

The particular problem of determining $u_{1m}(x, t)$ and $z(x, t)$ was solved with the aid of the integral equations of momentum and energy [1] with a specific scale h selected. In a second variant of calculations the integral relation for energy, with a rather intricate mathematical structure, was replaced with one of the expansions of H_i corresponding to that scale and set equal to the matching constant in series (9). In the equations to be solved we had retained only the terms containing parameters with not higher than first-order derivatives of u_{1m} and z with respect to x and t.

Problems of buildup and decay of an inundated jet with an exponentially varying momentum $I \sim e^{\pm 2t}$ were also taken into consideration. An analogous but differently formulated problem was dealt with by L. A. Bulis et al. [3]. In order to make it possible to compare our solution with theirs, we stipulated the initial conditions for u_{1m} and z in accordance with the results of their study [3].

The systems of two first-order partial differential equations were reduced to dimensionless form for integration on a "Nairi-3" computer, with the original system approximated according to the "running" count scheme of first-order accuracy so convenient for solving a mixed Cauchy problem.

The graphs in Fig. 1 depict the results of integration, namely the dimensionless axial jet velocity $\tilde{u}_{1m} = u_{1m}/u_{1m0}$ and characteristic jet thickness $\tilde{z} = z/z_0$ as functions of the dimensionless space coordinate $\tilde{x} = (x - x_0)/u_{1m0} z_0$ and time $\tilde{t} = (t - t_0)/z_0$ ("0" in the superscript refers to the initial instant of jet discharge). Alongside the curves from study [3] there are shown here curves depicting the relations based on the solution to two systems of equations: (a) integral equations of momentum and energy, (b) integral equation of momentum and expansion (10) of the characteristic function H_1 equal to a constant ($H_1 = 0.943$). The main trends in development of transient inundated jets are the same according to the different methods of calculation. A jet narrows during buildup, its maximum velocity at fixed points on the axis increasing with time. A jet widens during decay, its velocity at a fixed point on the axis decreasing with time. The basic differences existing between our solution and the solution in study [3] are attributable to the fact that both methods are approximate, their truthfulness being verifiable only through comparison with experimental data. No such experimental data pertaining to laminar jets have been published so far.

NOTATION

x, longitudinal coordinate; y, transverse coordinate; t, time; η, dimensionless coordinate; U(x, t), velocity of the concurrent stream; ψ, flow function; φ, dimensionless flow function; u_{1m}, axial excess velocity in the jet; h, scale of the transverse coordinate; $z = h^2/\nu$; ν, kinematic viscosity; K, L, M, N, P, Q, functionals; f_{kn}, r_{ij}, g_{lm}, parameters; δ_i, characteristic jet thicknesses; H_i, characteristic functions; B, normalizing constant; and β, constant in the Schlichting solution.

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